The Jacobi-Rosochatius problem on an ellipsoid: the Lax representations and billiards

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ABSTRACT. The Lax representations of the geodesic flow, the Jacobi-Rosochatius problem and its perturbations by means of separable polynomial potentials, on an ellipsoid are constructed. We prove complete integrability in the case of a generic symmetric ellipsoid and describe analogous systems on complex projective spaces. Also, we consider billiards within an ellipsoid under the influence of the Hook and Rosochatius potentials between the impacts. A geometric interpretation of the integrability analogous to the classical Chasles and Poncelet theorems is given.

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1. Introduction

A well known Jacobi's problem describes the motion of a material point on a n-dimensional ellipsoid E^n

$$(1.1) \qquad E^n = \{x \in \mathbb{R}^{n+1} \,|\, \langle A^{-1}x, x \rangle = 1\}$$

under the influence of the Hook elastic force $-\sigma x$. Here $A = \mathrm{diag}(a_0, a_1, \ldots, a_n)$ is a positive definite matrix and σ is a real parameter [24, 36]. The Lagrangian of the

system is $L(x, \dot{x}) = \frac{1}{2}\langle \dot{x}, \dot{x}\rangle - \frac{\sigma}{2}\langle x, x\rangle$. The corresponding Euler-Lagrange equations are

$$(1.2) \qquad \ddot{x} = \lambda A^{-1}x - \sigma x,$$

where the Lagrange multiplier is $\lambda = -(\langle A^{-1}\dot{x}, \dot{x}\rangle - \sigma)/\langle A^{-2}x, x\rangle$.

By introducing the momenta $y = \partial L/\partial \dot{x} = \dot{x}$, we can write (1.2) in the Hamiltonian form

$$(1.3) \qquad \dot{x} = y, \qquad \dot{y} = -\frac{\langle A^{-1}y, y\rangle - \sigma}{\langle A^{-2}x, x\rangle}A^{-1}x - \sigma x$$

on the cotangent bundle T^*E^n realized by the constraints

$$(1.4) \qquad F_1 = \langle A^{-1}x, x\rangle - 1 = 0, \qquad F_2 = \langle A^{-1}x, y\rangle = 0.$$

The equations (1.3) are Hamiltonian with respect to the Hamiltonian

$$(1.5) \qquad H(x, y) = \langle y, \dot{x}\rangle - L(x, \dot{x})|_{\dot{x}=y} = \frac{1}{2}\langle y, y\rangle + \frac{\sigma}{2}\langle x, x\rangle$$

and the Dirac-Poisson bracket defined by

$$(1.6) \qquad \{f_1, f_2\}_D = \{f_1, f_2\} - \frac{\{F_1, f_1\}\{F_2, f_2\} - \{F_2, f_1\}\{F_1, f_2\}}{\{F_1, F_2\}},$$

where $\{\cdot, \cdot\}$ is the canonical Poisson bracket on $\mathbb{R}^{2n+2}(x, y)$ (see [**3, 36**]). The Dirac-Poisson bracket of the coordinate functions are:

$$\{x_i, x_j\}_D = 0, \quad \{y_i, y_j\}_D = -\frac{x_i y_j - x_j y_i}{a_i a_j \langle A^{-2}x, x\rangle}, \quad \{x_i, y_j\}_D = \delta_{ij} - \frac{x_i x_j}{a_i a_j \langle A^{-2}x, x\rangle}.$$

Note that the Hamiltonian flow and the Dirac-Poisson structure is defined not only on T^*E^n but on the whole $\mathbb{R}^{2n+2}$ without $x = 0$.

If we set $\sigma = 0$, the system represents the geodesic flow on the ellipsoid (1.1). The geodesic flow and the Jacobi problem are one of the basic classical models of integrable systems [**24, 3**]. For the ellipsoids with distinct semi-axes, they are separable in the Jacobi elliptic coordinates $(\lambda_0 = 0, \lambda_1, \ldots, \lambda_n)$ defined as follows [**3, 36, 37, 50**]. Through every point $x \in \mathbb{R}^{n+1}$, $x_0 \cdot x_1 \cdot \cdots \cdot x_n \neq 0$, pass exactly $n+1$ mutually orthogonal confocal quadrics $\mathcal{Q}_{\lambda_0}, \mathcal{Q}_{\lambda_1}, \ldots, \mathcal{Q}_{\lambda_n}$ $(\lambda_0 < a_0 < \lambda_1 < a_1 < \cdots < \lambda_n < a_n)$ given by the equation

$$(1.7) \qquad \mathcal{Q}_\lambda: \qquad \langle (A - \lambda)^{-1}x, x\rangle = \sum_{i=0}^{n} \frac{x_i^2}{a_i - \lambda} = 1.$$

The original coordinates are defined up to a sign:

$$x_k^2 = \frac{\prod_i (a_k - \lambda_i)}{\prod_{i, i\neq k}(a_k - a_i)}, \quad k = 0, \ldots, n.$$

In his celebrated paper [**36**], Moser found the Lax representation $\dot{L} = [L, B]$,

$$L = \left(\mathbf{I}_{n+1} - \frac{y \otimes y}{\langle y, y\rangle}\right)(A - x \otimes x)\left(\mathbf{I}_{n+1} - \frac{y \otimes y}{\langle y, y\rangle}\right), \quad B = A^{-1}x \wedge A^{-1}y,$$

of the associated system

$$(1.8) \qquad \dot{x} = \frac{\partial \Phi}{\partial y}, \quad \dot{y} = -\frac{\partial \Phi}{\partial x}, \quad \Phi = -\langle y, A^{-1}y \rangle (1 - \langle x, A^{-1}x \rangle) - \langle x, A^{-1}y \rangle^2.$$

The set $\Phi = 0$ describes the set of tangents of the ellipsoid $E^n = \mathcal{Q}_0$. Restricted to $\Phi = 0$, the system (1.8) has the following important property: the moving line $p(t) = \{x(t) + sy(t) \,|\, s \in \mathbb{R}\}$ has a point of contact ξ with E^n along nonparameterized geodesic $\xi(\tau)$. Moreover, the eigenvalues $\eta_1, \dots, \eta_n$ of L (different from zero), define n quadrics $\mathcal{Q}_{\eta_1}, \dots, \mathcal{Q}_{\eta_n}$ from the confocal family (1.7), such that $p(t)$ is simultaneously tangent to $\mathcal{Q}_{\eta_1}, \dots, \mathcal{Q}_{\eta_n}$ (a variant of the classical Chasles's theorem [8]).

Although Moser's method is applied in the construction of various integrable models admitting Lax representations [1], Lax representations with a spectral parameter of the geodesic flow and the Jacobi problem on E^n, according to author's knowledge, are not given yet.

In Section 2 we define double Jacobi flows and complex double Jacobi flows and construct two different Lax representations, a "small" 2×2 one (Theorem 2.1) and a "big" $(n+1) \times (n+1)$ one (Theorem 2.2). An appropriate restrictions of the double Jacobi flows and the complex double Jacobi flows are the Jacobi problems on the ellipsoids (1.1) and

$$(1.9) \qquad\qquad E^{2n+1} = \{z \in \mathbb{C}^{n+1} \,|\, \langle A^{-1}z, \bar{z} \rangle = 1\},$$

respectively, leading to the Lax representations of the Jacobi problem (Theorem 3.1, Section 3). Note that a small 2×2 Lax representation has the usual Lax matrix form of the *Jacobi-Mamford* system (see [33, 45]).

The Jacobi problem on the ellipsoid (1.9) is invariant with respect to the standard $\mathbb{T}^{n+1}$-action on $\mathbb{C}^{n+1}$.[1] It is well known (e.g., see [36, 28]), that the reduced flow can be naturally considered as a system describing the motion of a material point on the ellipsoid (1.1) under the influence of the Hook and the Rosochatius potentials [40]

$$(1.10) \qquad\qquad V(x) = \frac{\sigma}{2} \langle x, x \rangle + \frac{1}{2} \sum_{k=0}^{n} \frac{\mu_k^2}{x_k^2}$$

(Section 4). The Lax representation is also invariant with respect to the standard $\mathbb{T}^{n+1}$-action and induce a small 2×2-Lax representation with a spectral parameter for the Jacobi-Rosochatius system on the ellipsoid E^n (Theorem 4.1). Note that if instead of the $\mathbb{T}^{n+1}$-reduction, we perform a S^1-reduction, with S^1 diagonally embedded in $\mathbb{T}^{n+1}$, we obtain a natural mechanical system on a complex projective space (Proposition 4.1), providing a class of examples of Hermite-Liouville manifolds (see Remark 4.1) [23, 44].

If all semi-axes of the ellipsoid (1.1) are distinct, the Jacobi-Rosochatius system is separable in elliptic coordinates and has n independent commuting integrals

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[1]We use the complex notation to simplify the reduction procedure. Equivalently one can consider the real space $\mathbb{R}^{2n+2}$ and the ellipsoid E^{2n+1} defined by the matrix $A = (a_0, a_0, a_1, a_1, \dots, a_n, a_n)$.

which are quadratic in momenta. The geodesic flows on symmetric ellipsoids are studied in details in [**10**]. In general, the geodesic flows are integrable in a noncommutative sense. In Section 5, we prove complete integrability of the symmetric Jacobi-Rosochatius system both in a noncommutative (Theorem 5.2) and the Liouville sense (Theorem 5.3).

In Section 6 we consider the billiard system within an ellipsoid under the influence of the potential (1.10) between the impacts. By a slight modification of the results given by Fedorov [**21**], we describe the billiard mapping and the Lax representation (Theorem 6.1). A geometric interpretation of the integrability analogous to classical Chasles and Poncelet theorems is given (Theorem 6.2).

Finally, in the last section, by using the 2×2 Lax representation, we give the hierarchy of the Lax representations for the separable potential perturbations of the Jacobi-Rosochatius system on the ellipsoid E^n.

2. Double Jacobi flows

Together with the Jacobi problem, let us consider a system defined by the Lagrangian $L(x, \xi, \dot{x}, \dot{\xi}) = \langle \dot{x}, \dot{\xi} \rangle - \sigma \langle x, \xi \rangle$ and constrained on the hypersurface

$$\Sigma = \{(x, \xi) \in \mathbb{R}^{2n+2} \,|\, \langle x, A^{-1}\xi \rangle - 1 = 0\}.$$

The Euler-Lagrange equations are

$$\ddot{x} = \lambda A^{-1}x - \sigma x,$$
$$\ddot{\xi} = \lambda A^{-1}\xi - \sigma \xi,$$

where the Lagrange multiplier is

$$\lambda = -\frac{\langle A^{-1}\dot{x}, \dot{\xi} \rangle - \sigma}{\langle A^{-2}x, \xi \rangle}.$$

As in the case of the Jacobi problem, we can write the corresponding Hamiltonian equations on $T^*\Sigma$. Canonically conjugate momenta to (x, ξ) are $\eta = \partial L/\partial \dot{x} = \dot{\xi}$, $y = \partial L/\partial \dot{\xi} = \dot{x}$. The Hamiltonian function of the system is given by the Legendre transformation:

$$H(x, \xi, \eta, y) = \langle \dot{x}, \eta \rangle + \langle \dot{\xi}, y \rangle - L = \langle y, \eta \rangle + \sigma \langle x, \xi \rangle,$$

and the Hamiltonian equations are

$$(2.1) \qquad \dot{x} = y, \qquad \dot{y} = -\frac{\langle A^{-1}y, \eta \rangle - \sigma}{\langle A^{-2}x, \xi \rangle} A^{-1}x - \sigma x,$$

$$(2.2) \qquad \dot{\xi} = \eta, \qquad \dot{\eta} = -\frac{\langle A^{-1}y, \eta \rangle - \sigma}{\langle A^{-2}x, \xi \rangle} A^{-1}\xi - \sigma \xi.$$

The symplectic structure on

$$(2.3) \qquad T^*\Sigma: \quad G_1 = \langle x, A^{-1}\xi \rangle - 1 = 0, \quad G_2 = \langle y, A^{-1}\xi \rangle + \langle x, A^{-1}\eta \rangle = 0$$

is the restriction of the standard symplectic structure on $\mathbb{R}^{4n+4}(x, \xi, \eta, y)$. The corresponding Dirac-Poisson structure is given by (1.6), where we should replace the

constraint functions F_1, F_2 by G_1, G_2 and $\{\cdot, \cdot\}$ is the canonical Poisson structure on $\mathbb{R}^{4n+4}(x, \xi, \eta, y)$.

By the analogy with the double Neumann system (see [41]), we refer to (2.1), (2.2) as a *double Jacobi flow*.

We can also consider the complexified phase space, an affine $2n+2$-dimensional variety in $\mathbb{C}^{4n+4}(x, \xi, \eta, y)$ defined by (2.3). Then we refer to (2.1), (2.2) as a *complex double Jacobi flow* (here, the time parameter t can be both: real or complex).

Let $A_\lambda = \operatorname{diag}(\lambda - a_0, \lambda - a_1, \dots, \lambda - a_n)$ and

$$q_\lambda(x, \xi) = \langle A_\lambda^{-1} x, \xi \rangle = \sum_{i=0}^{n} \frac{x_i \xi_i}{\lambda - a_i}.$$

THEOREM 2.1. *The (complex) double Jacobi flow* (2.1), (2.2) *implies the matrix equation*

(2.4) $$\dot{\mathcal{L}}(\lambda) = [\mathcal{L}(\lambda), \mathcal{A}(\lambda)]$$

with 2×2 matrices depending on the parameter λ

$$\mathcal{L}(\lambda) = \begin{pmatrix} q_\lambda(x, \eta) & q_\lambda(y, \eta) + \sigma \\ -1 - q_\lambda(x, \xi) & -q_\lambda(y, \xi) \end{pmatrix},$$

$$\mathcal{A}(\lambda) = \frac{1}{\langle A^{-2} x, \xi \rangle} \begin{pmatrix} 0 & \frac{1}{\lambda}(\sigma - \langle A^{-1} y, \eta \rangle) - \sigma \langle A^{-2} x, \xi \rangle \\ \langle A^{-2} x, \xi \rangle & 0 \end{pmatrix}.$$

PROOF. To simplify the calculations, consider the time reparametrization

(2.5) $$dt = \langle A^{-2} x, \xi \rangle d\tau.$$

The double Jacobi flow in the new time gets the symmetric form

(2.6)
$$\begin{aligned}
x' &= \langle A^{-2} x, \xi \rangle y, \\
y' &= (\sigma - \langle A^{-1} y, \eta \rangle) A^{-1} x - \sigma \langle A^{-2} x, \xi \rangle x, \\
\xi' &= \langle A^{-2} x, \xi \rangle \eta, \\
\eta' &= (\sigma - \langle A^{-1} y, \eta \rangle) A^{-1} \xi - \sigma \langle A^{-2} x, \xi \rangle \xi,
\end{aligned}$$

where $(\cdot)' = \frac{d}{d\tau}(\cdot) = \langle A^{-2} x, \xi \rangle \frac{d}{dt}(\cdot)$.

By using the constraints (2.3) and the identity

$$A^{-1} A_\lambda^{-1} = A_\lambda^{-1} A^{-1} = (A^{-1} + A_\lambda^{-1})/\lambda,$$

we obtain the relations

$$\begin{aligned}
(q_\lambda(x, \eta))' = (q_\lambda(y, \xi))' &= \frac{1}{\lambda}(1 + \langle A_\lambda^{-1} x, \xi \rangle)(\sigma - \langle A^{-1} y, \eta \rangle) \\
&\quad + \langle A^{-2} x, \xi \rangle(\langle y, A_\lambda^{-1} \eta \rangle - \sigma \langle x, A_\lambda^{-1} \xi \rangle), \\
(q_\lambda(y, \eta))' &= \frac{1}{\lambda}(\langle A_\lambda^{-1} x, \eta \rangle + \langle A_\lambda^{-1} y, \xi \rangle)(\sigma - \langle A^{-1} y, \eta \rangle) \\
&\quad - \sigma \langle A^{-2} x, \xi \rangle(\langle x, A_\lambda^{-1} \eta \rangle + \langle A_\lambda^{-1} y, \xi \rangle), \\
(q_\lambda(x, \xi))' &= \langle A^{-2} x, \xi \rangle(\langle x, A_\lambda^{-1} \eta \rangle + \langle A_\lambda^{-1} y, \xi \rangle),
\end{aligned}$$

Now, the reader can simply verify $\mathcal{L}' = [\mathcal{L}, \langle A^{-2} x, \xi \rangle \mathcal{A}]$. $\qquad \square$

Therefore, the coefficients of the invariant polynomials of the matrix $a(\lambda)\mathcal{L}(\lambda)$ are the first integrals of the system. If all of a_i are distinct, the integrals can be written in the form

$$f_i = y_i\eta_i + \sigma x_i\xi_i + \sum_{j\neq i}\frac{(y_ix_j - y_jx_i)(\eta_i\xi_j - \eta_j\xi_i)}{a_i - a_j},$$
$$g_i = y_i\xi_i - x_i\eta_i, \qquad i = 0, 1, \ldots, n.$$

THEOREM 2.2. *The (complex) double Jacobi flow* (2.1), (2.2) *restricted to the invariant variety*

$$(2.7) \qquad\qquad \langle A^{-1}x, \eta\rangle = \langle A^{-1}y, \xi\rangle = 0.$$

implies the matrix equation

$$(2.8) \qquad\qquad \dot{\mathcal{L}}^*(\lambda) = [\mathcal{A}^*(\lambda), \mathcal{L}^*(\lambda)]$$

with $(n+1) \times (n+1)$ *matrices depending on the parameter* λ

$$\mathcal{L}^*(\lambda) = \lambda(y\otimes\xi - x\otimes\eta) + y\otimes\eta + \sigma x\otimes\xi - \sigma A - \lambda^2 A,$$
$$\mathcal{A}^*(\lambda) = \frac{1}{\langle A^{-2}x, \xi\rangle}\left(A^{-1}y\otimes A^{-1}\xi - A^{-1}x\otimes A^{-1}\eta + \lambda A^{-1}\right).$$

PROOF. From (2.6) we get

$$(x\otimes\xi)' = \langle A^{-2}x, \xi\rangle(y\otimes\xi + x\otimes\eta),$$
$$(y\otimes\eta)' = (\sigma - \langle A^{-1}y, \eta\rangle)(A^{-1}x\otimes\eta + y\otimes A^{-1}\xi) - \sigma\langle A^{-2}x, \xi\rangle(x\otimes\eta + y\otimes\xi),$$
$$(y\otimes\xi - x\otimes\eta)' = (\sigma - \langle A^{-1}y, \eta\rangle)(A^{-1}x\otimes\xi - x\otimes A^{-1}\xi).$$

Thus,

$$(\mathcal{L}^*(\lambda))' = (\sigma - \langle A^{-1}y, \eta\rangle)\left(\lambda(A^{-1}x\otimes\xi - x\otimes A^{-1}\xi) + A^{-1}x\otimes\eta + y\otimes A^{-1}\xi\right).$$

Further, let us denote

$$\Omega = A^{-1}y\otimes A^{-1}\xi - A^{-1}x\otimes A^{-1}\eta.$$

Then, by using the constraints (2.3), we get

$$[\Omega, y\otimes\eta] = \langle A^{-1}y, \xi\rangle A^{-1}y\otimes\eta + \langle A^{-1}x, \eta\rangle y\otimes A^{-1}\eta$$
$$\qquad -\langle A^{-1}\eta, y\rangle(A^{-1}x\otimes\eta + y\otimes A^{-1}\xi),$$
$$[\Omega, x\otimes\xi] = A^{-1}y\otimes\xi - \langle A^{-1}\eta, x\rangle A^{-1}x\otimes\xi - \langle A^{-1}y, \xi\rangle x\otimes A^{-1}\xi + x\otimes A^{-1}\eta,$$
$$[\Omega, y\otimes\xi - x\otimes\eta] = y\otimes A^{-1}\eta - A^{-1}y\otimes\eta + \langle A^{-1}x, \eta\rangle(A^{-1}x\otimes\eta - x\otimes A^{-1}\eta)$$
$$\qquad +\langle A^{-1}y, \xi\rangle(A^{-1}y\otimes\xi - y\otimes A^{-1}\xi) - \langle A^{-1}y, \eta\rangle(A^{-1}x\otimes\xi - x\otimes A^{-1}\xi),$$
$$[\Omega, A] = A^{-1}y\otimes\xi - A^{-1}x\otimes\eta - y\otimes A^{-1}\xi + x\otimes A^{-1}\eta,$$

implying

$$[\langle A^{-2}x,\xi\rangle \mathcal{A}^*(\lambda), \mathcal{L}^*] = [\Omega + \lambda A^{-1}, \lambda(y\otimes\xi - x\otimes\eta) + y\otimes\eta + \sigma x\otimes\xi - \sigma A - \lambda^2 A]$$
$$= (\sigma - \langle A^{-1}y,\eta\rangle)\left(\lambda(A^{-1}x\otimes\xi - x\otimes A^{-1}\xi) + A^{-1}x\otimes\eta + y\otimes A^{-1}\xi\right)$$
$$+\langle A^{-1}y,\xi\rangle(A^{-1}y\otimes\eta - \sigma x\otimes A^{-1}\xi) + \langle A^{-1}x,\eta\rangle(y\otimes A^{-1}\eta - \sigma A^{-1}x\otimes\xi)$$
$$+\lambda\left(\langle A^{-1}x,\eta\rangle(A^{-1}x\otimes\eta - x\otimes A^{-1}\eta) + \langle A^{-1}y,\xi\rangle(A^{-1}y\otimes\xi - y\otimes A^{-1}\xi)\right).$$

It remains to note that (2.7) defines an invariant manifold of the system. Indeed, we have

$$(\langle A^{-1}x,\eta\rangle)' = \langle A^{-2}x,\xi\rangle\langle y, A^{-1}\eta\rangle + (\sigma - \langle A^{-1}y,\eta\rangle)\langle A^{-2}x,\xi\rangle - \sigma\langle A^{-2}x,\xi\rangle = 0.$$

$\square$

Note that the matrix $\mathcal{L}(\lambda)$ is invariant under the transformations of the phase space given by the $(\mathbb{R}^*)^{n+1}$-action (respectively $(\mathbb{C}^*)^{n+1}$-action):

$$(x_i, \xi_i, \eta_i, y_i) \longmapsto (s_i x_i, s_i^{-1}\xi_i, s_i^{-1}\eta_i, s_i y_i), \qquad s_i \neq 0, \qquad i = 0, 1, \ldots, n,$$

while the matrix $\mathcal{L}^*(\lambda)$ is $\mathbb{R}^*$-invariant (respectively $\mathbb{C}^*$-invariant):

$$(x_i, \xi_i, \eta_i, y_i) \longmapsto (s x_i, s^{-1}\xi_i, s^{-1}\eta_i, s y_i), \qquad s \neq 0, \qquad i = 0, 1, \ldots, n.$$

3. The Lax representations of the Jacobi problem

The equations

$$x = \xi, \qquad y = \eta$$

define the invariant manifold of (2.1), (2.2), so the double Jacobi flow contains as a subsystem the Jacobi problem (1.3). Also, for $x = \xi$, $y = \eta$, the condition (2.7) is satisfied and the above theorems imply Lax representations for the Jacobi problem. In particular, when we set $\sigma = 0$, we get the Lax representations for the geodesic flow on a ellipsoid.

THEOREM 3.1. *The Jacobi problem* (1.3) *implies the matrix equations*

(3.1) $$\dot{\mathcal{L}}(\lambda) = [\mathcal{L}(\lambda), \mathcal{A}(\lambda)]$$

and

(3.2) $$\dot{\mathcal{L}}^*(\lambda) = [\mathcal{A}^*(\lambda), \mathcal{L}^*(\lambda)]$$

with 2×2 *and* $(n+1) \times (n+1)$ *matrices depending on the parameter* λ

$$\mathcal{L}(\lambda) = \begin{pmatrix} q_\lambda(x,y) & q_\lambda(y,y) + \sigma \\ -1 - q_\lambda(x,x) & -q_\lambda(y,x) \end{pmatrix},$$

$$\mathcal{A}(\lambda) = \frac{1}{\langle A^{-2}x,x\rangle}\begin{pmatrix} 0 & \frac{1}{\lambda}(\sigma - \langle A^{-1}y,y\rangle) - \sigma\langle A^{-2}x,x\rangle \\ \langle A^{-2}x,x\rangle & 0 \end{pmatrix},$$

$$\mathcal{L}^*(\lambda) = \lambda(y\wedge x) + y\otimes y + \sigma x\otimes x - \sigma A - \lambda^2 A,$$

$$\mathcal{A}^*(\lambda) = \frac{1}{\langle A^{-2}x,x\rangle}\left(A^{-1}y\wedge A^{-1}x + \lambda A^{-1}\right),$$

respectively.

From the Lax representations we obtain the well known form of the integrals of the Jacobi problem given by Moser [36]. Note that the 2×2 Lax matrix $\mathcal{L}(\lambda)$ has the usual Lax matrix form of the *Jacobi-Mamford* systems (see [33, 45]). Also note, if all a_i are distinct, the Lax representations (3.1) and (3.2) are equivalent to the Jacobi problem (1.3), up to the action of groups $\mathbb{Z}_2^{n+1}$, $\mathbb{Z}_2$ generated by reflections

$$(x_i, y_i) \longmapsto (s_i x_i, s_i y_i), \quad s_i = \pm 1, \qquad i = 0, 1, \dots, n,$$
$$(x, y) \longmapsto s(x, y), \qquad s = \pm 1.$$

The time reperemetrezation (2.5), for $x = \xi$, coincides with the time reparametrization $dt = \lambda_1 \dots \lambda_n d\tau$ used in the integration of the geodesic flow [48, 5, 20].

4. The Jacobi-Rosochatius problem on an ellipsoid

Consider the complex double Jacobi flow (2.1), (2.2) on the invariant real variety

$$(4.1) \qquad x = z, \quad \xi = \bar{z}, \quad y = p, \quad \eta = \bar{p}.$$

The equations (2.3), (4.1) define the cotangent bundle $T^* E^{2n+1}$ of the $2n + 1$-dimensional real ellipsoid in the complex space (1.9)

$$(4.2) \qquad T^* E^{2n+1}: \quad \langle A^{-1} z, \bar{z} \rangle - 1 = 0, \quad \langle A^{-1} z, \bar{p} \rangle + \langle A^{-1} \bar{z}, p \rangle = 0.$$

The complex double Jacobi flow restricted to $T^* E^{2n-1}$

$$(4.3) \qquad \dot{z} = p, \qquad \dot{p} = -\frac{\langle A^{-1} p, \bar{p} \rangle - \sigma}{\langle A^{-2} z, \bar{z} \rangle} A^{-1} z - \sigma z,$$

describes the motion of a material point on E^{2n+1}, under the influence of the elastic force $-\sigma z$. From Theorem 2.1 we see that the system (4.3) implies the matrix equation

$$(4.4) \qquad \dot{\mathcal{L}}(\lambda) = [\mathcal{L}(\lambda), \mathcal{A}(\lambda)],$$

where

$$\mathcal{L}(\lambda) = \begin{pmatrix} q_\lambda(z, \bar{p}) & q_\lambda(p, \bar{p}) + \sigma \\ -1 - q_\lambda(z, \bar{z}) & -q_\lambda(p, \bar{z}) \end{pmatrix},$$
$$\mathcal{A}(\lambda) = \frac{1}{\langle A^{-2} z, \bar{z} \rangle} \begin{pmatrix} 0 & \frac{1}{\lambda}(\sigma - \langle A^{-1} p, \bar{p} \rangle) - \sigma \langle A^{-2} z, \bar{z} \rangle \\ \langle A^{-2} z, \bar{z} \rangle & 0 \end{pmatrix}.$$

The system (4.3), as well as the Lax representation (4.4), is invariant with respect to the Hamiltonian torus action on $T^* E^{2n+1}$:

$$(4.5) \qquad (z_k, p_k) \longmapsto e^{i\varphi_k}(z_k, p_k), \qquad k = 0, 1, \dots, n,$$

with the momentum mapping

$$(4.6) \qquad \mathbf{h} = (h_0, h_1, \dots, h_n), \qquad h_k = -\frac{i}{2} g_k = \frac{i}{2}(z_k \bar{p}_k - p_k \bar{z}_k).$$

4.1. The Jacobi system on a complex projective space. In particular, the Hamiltonian $h = h_0 + \cdots + h_n$ induces a S^1-action

$$(4.7) \qquad (z, p) \mapsto e^{i\varphi}(z, p).$$

The symplectic reduced space $h^{-1}(0)/S^1$ is simplectomorphic to the cotangent bundle of the complex projective space $\mathbb{P}^n$. The reduced system is a natural mechanical system with the kinetic energy determined by the ellipsoidal metric on $\mathbb{P}^n$, the submersion metric with respect to S^1-action, under the influence of the "elastic" force. From Theorems 5.2 and 5.3, where we identify $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ and set $A = \mathrm{diag}(a_0, a_0, a_1, a_1, \ldots, a_n, a_n)$, we have integrability of the Jacobi problem (4.2) for an arbitrary choice of a_i. Similarly, the reduced flow is integrable. A general treatment of the integrability of the reduced systems is given in [**52, 25**]. Here, the orbit of the S^1-action are tangent to the isotropic tori $\mathcal{T}$, which lay in $h^{-1}(0)$, and $\mathcal{T}/S^1$ are invariant isotropic tori for the reduced flow.

In the next statement we describe the reduced system on $\mathbb{P}^n$. Let $\pi : \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n$ be the canonical projection of $w = (w_0, \ldots, w_n) \in \mathbb{C}^{n+1} \smallsetminus \{0\}$ to $[w] = [w_0 : \cdots : w_n] \in \mathbb{P}^n$, with respect to the C^*-action.

PROPOSITION 4.1. *The reduced Jacobi problem on $\mathbb{P}^n$ is a natural mechanical system with the kinetic energy determined by the metric*

$$\tilde{g}_A(\pi_* X, \pi_* X)|_{[w]} = \frac{\langle w, A\bar{w}\rangle \langle X, A\bar{X}\rangle - \langle X, A\bar{w}\rangle \langle w, A\bar{X}\rangle}{\langle w, A\bar{w}\rangle \langle w, \bar{w}\rangle}, \quad X \in T_w \mathbb{C}^{n+1} \smallsetminus \{0\}$$

under the influence of the potential field $V([w]) = \sigma\langle w, A\bar{w}\rangle / 2\langle w, \bar{w}\rangle$.

PROOF. Under the change of variables $z_i = \sqrt{a_i} w_i$, the Jacobi problem transforms to the system on a sphere $S^{2n+1} : \langle w, \bar{w}\rangle = 1$ with the Lagrangian function

$$L(w, \dot{w}) = \frac{1}{2}\langle \dot{w}, A\dot{\bar{w}}\rangle - \frac{\sigma}{2}\langle w, A\bar{w}\rangle.$$

It is well known that the reduced system on $h^{-1}(0)/S^1 \cong T^*\mathbb{P}^n$ can be described by the S^1-Lagrange-Routh reduction with a zero value of the S^1-momentum mapping (e.g., see [**3**]). Since it is convenient to work with homogeneous coordinates, we firstly extend L to the C^*-invariant Lagrangian function

$$(4.8) \qquad L = \frac{\langle \dot{w}, A\dot{\bar{w}}\rangle}{2\langle w, \bar{w}\rangle} - \frac{\sigma \langle w, A\bar{w}\rangle}{2\langle w, \bar{w}\rangle},$$

defined on $\mathbb{C}^{n+1} \smallsetminus \{0\}$ and then perform the Lagrange-Routh reduction with respect to the C^*-action.

Obviously, the reduced potential is $V([w]) = \sigma\langle w, A\bar{w}\rangle / 2\langle w, \bar{w}\rangle$. Further, the Lagrangian (4.8) defines the Riemannian and Hermitian metrics

$$g_A(X, Y) = \frac{\langle X, A\bar{Y}\rangle + \langle \bar{X}, AY\rangle}{2\langle w, \bar{w}\rangle}, \quad h_A(X, Y) = \frac{\langle X, A\bar{Y}\rangle}{\langle w, \bar{w}\rangle}, \quad X, Y \in T_w \mathbb{C}^{n+1} \smallsetminus \{0\},$$

respectively. The reduced system is a natural mechanical system $(\mathbb{P}^n, \tilde{g}_A, V([w]))$, where $\tilde{g}_A$ is the submersion metric with respect to the C^*-action [**3**].

At every $w \in \mathbb{C}^{n+1} \smallsetminus \{0\}$ we have a decomposition

$$T_w \mathbb{C}^{n+1} \smallsetminus \{0\} = \mathbb{C}^{n+1} = \mathcal{V}_w \oplus \mathcal{H}_w,$$

where $\mathcal{V}_w$ is the tangent space of the orbit of C^*-action through w (*vertical space at* w) and $\mathcal{H}_w$ is its g_A-orthogonal complement (*horizontal space at* w). Since $\mathcal{V}_w$ is the complex line through w, its real g_A-orthogonal complement coincides with its h_A-Hermitian orthogonal complement

$$\mathcal{H}_w = \{X \,|\, h_A(X, w) = 0\}.$$

Let $X, Y \in T_w \mathbb{C}^{n+1} \smallsetminus \{0\}$. By definition, the submersion metric $\tilde{g}_A(\pi_* X, \pi_*)|_{[w]}$ on $\mathbb{P}^n$ by is equal to $g_A(X', Y')|_w$, where X' and Y' are horizontal components of X and Y

$$X' = X - \frac{h_A(X, w)}{h_A(w, w)} w, \quad Y' = Y - \frac{h_A(Y, w)}{h_A(w, w)} w.$$

Therefore

$$
\begin{aligned}
\tilde{g}_A(\pi_* X, \pi_* X)|_{[w]} &= g_A(X', X')|_w = h_A(X', X')|_w = h_A(X', X) \\
&= h_A(X, X) - \frac{h_A(w, X) h_A(X, w)}{h_A(w, w)} \\
&= \frac{\langle w, A\bar{w} \rangle \langle X, A\bar{X} \rangle - \langle X, A\bar{w} \rangle \langle w, A\bar{X} \rangle}{\langle w, A\bar{w} \rangle \langle w, \bar{w} \rangle}
\end{aligned}
$$

$\square$

REMARK 4.1. Note that $(\mathbb{P}^n, \tilde{g}_A)$ is an example of a Hermite-Liouville manifold (see [**23**, **44**]). In particular, if A is the identity matrix, then $\tilde{g}_A$ is the standard Fubini-Study metric on $\mathbb{P}^n$. The integrability of the geodesic flow of the Fubini-Study metric is proved by Thimm [**43**] and Boyer, Kalnins and Winternitz [**7**]. Further, besides the Hook potential in (4.3), we can add other separable polynomial potentials $V(z)$ (see the last section), and by the Maupertuis principle (e.g., see [**3**]), $(\mathbb{P}^n, (c - V([w]))\tilde{g}_A)$, $c > \max V([w])$ are examples of Hermite-Liouville manifolds as well.

4.2. Reduction to the Jacobi-Rosochatius problem. Now we shall perform the reduction with respect to the torus action (4.5). It is well known that we obtain a natural mechanical system under the influence of the Rosochatius potential [**40**, **36**, **28**].

Introduce the canonical polar coordinates $(x_k, \varphi_k, y_k, h_k)$, where h_k are given by (4.6) and φ_k, y_k by

$$(4.9) \qquad z_k = x_k e^{i\varphi_k}, \quad p_k = y_k e^{i\varphi_k} + i\frac{h_k}{x_k} e^{i\varphi_k}, \qquad k = 0, 1, \dots, n.$$

The Hamiltonian of the Jacobi problem in new coordinates reads

$$H = \frac{1}{2} \langle p, \bar{p} \rangle + \frac{\sigma}{2} \langle z, \bar{z} \rangle = \frac{1}{2} \langle y, y \rangle + \frac{\sigma}{2} \langle x, x \rangle + \frac{1}{2} \sum_{k=0}^{n} \frac{h_k^2}{x_k^2}.$$

We see that φ_k are cyclic variables of the system. Consider the level set of the momentum mapping

$$(T^*E^{2n+1})_\mu : \qquad \mathbf{h} = (\mu_0, \mu_1, \ldots, \mu_n)$$

(some of constants μ_k may be equal to zero).

The equations of the system on $(T^*E^{2n+1})_\mu$ separate on *reconstruction equations*

$$x_k^2 \dot{\varphi}_k = \mu_k, \qquad k = 0, 1, \ldots, n.$$

and the *reduced system* on $(T^*E^{2n+1})_\mu / \mathbb{T}^{n+1}$ in variables (x, y):

$$(4.10) \qquad \dot{x}_k = y_k, \quad \dot{y}_k = -\frac{\langle A^{-1}y, y \rangle + \langle A^{-1}\frac{\mu}{x}, \frac{\mu}{x} \rangle - \sigma}{\langle A^{-2}x, x \rangle} a_k^{-1} x_k - \sigma x_k + \frac{\mu_k^2}{x_k^3},$$

where $\mu/x = (\mu_0/x_0, \mu_1/x_1, \ldots, \mu_n/x_n)$.

Note that, according to the definition (4.9), we have $x_k \geq 0$. Also, the variables (x, y) satisfy the constraints (1.4). However, we can consider the system (4.10) on the whole cotangent bundle (1.4). Then it represents the motion of a material point on a ellipsoid with the potential energy $V(x)$ having two terms: the Hook and the Rosochatius potential (1.10).

Applying the change of variables (4.9) to the Lax representation (4.4) on the invariant set $(T^*E^{2n+1})_\mu$, after subtracting the multiple of the identity matrix from $\mathcal{L}(\lambda)$, we get the following statement.

THEOREM 4.1. *Suppose that the eigenvalues a_i of the matrix A are distinct. Up to the action of the group $\mathbb{Z}_2^{n+1}$ generated by the reflections*

$$(x_i, y_i) \longmapsto (s_i x_i, s_i y_i), \quad s_i = \pm 1, \qquad i = 0, 1, \ldots, n,$$

the Jacobi-Rosochatius problem (4.10) is equivalent to the matrix equation

$$(4.11) \qquad\qquad \dot{\mathcal{L}}(\lambda) = [\mathcal{L}(\lambda), \mathcal{A}(\lambda)]$$

with 2×2 matrices depending on the parameter λ

$$\mathcal{L}(\lambda) = \begin{pmatrix} q_\lambda(x, y) & q_\lambda(y, y) + q_\lambda(\frac{\mu}{x}, \frac{\mu}{x}) + \sigma \\ -1 - q_\lambda(x, x) & -q_\lambda(y, x) \end{pmatrix},$$

$$\mathcal{A}(\lambda) = \frac{1}{\langle A^{-2}x, x \rangle} \begin{pmatrix} 0 & \frac{1}{\lambda}(\sigma - \langle A^{-1}y, y \rangle - \langle A^{-1}\frac{\mu}{x}, \frac{\mu}{x} \rangle) - \sigma\langle A^{-2}x, x \rangle \\ \langle A^{-2}x, x \rangle & 0 \end{pmatrix}.$$

The integrals obtained by (4.11) can be written in the form

$$(4.12) \qquad f_i = y_i^2 + \sigma x_i^2 + \frac{\mu_i^2}{x_i^2} + \sum_{j \neq i} \frac{1}{a_i - a_j} \left((y_i x_j - y_j x_i)^2 + \frac{\mu_i^2 x_j^2}{x_i^2} + \frac{\mu_j^2 x_i^2}{x_j^2} \right),$$

$i = 0, 1, \ldots, n$. This is a commutative set of functions, both with respect to the canonical Poisson brackets $\{\cdot, \cdot\}$ and the Dirac-Poisson bracket $\{\cdot, \cdot\}_D$.

Let us note that a possible alternative approach in the construction of the Lax representations for the Jacobi-Rosochatius problem (4.10) is by using the Lax representations of the Neumann problem (e.g., see [**41**]) and the well known correspondence between the Neumann problem and the geodesic flow on an ellipsoid

[**31**, **28**]. An algebro-geometric study of the Neumann system on a sphere S^n with the addition of the Rosochatius potential is given in [**18**].

5. Symmetric ellipsoids. Complete integrability

Consider the case of a symmetric ellipsoid:

$$
\begin{aligned}
&a_i = \alpha_0, &&i \in I_0 = \{0, \dots, k_0 - 1\}, \\
(5.1)\quad &a_i = \alpha_1, &&i \in I_1 = \{k_0, \dots, k_0 + k_1 - 1\}, \\
&\cdots \\
&a_i = \alpha_r, &&i \in I_r = \{k_0 + \cdots + k_{r-1}, \dots, k_0 + \cdots + k_{r-1} + k_r - 1\},
\end{aligned}
$$

$\alpha_i \neq \alpha_j, i \neq j$, $k_0 + \cdots + k_r = n + 1$. Then the Jacobi-Rosochatius problem (4.10) is not equivalent to the Lax representation (4.11), but still implies it. From the Lax representation (4.11), the invariants of $\mathcal{L}(\lambda)$ are integrals of the flow. By using the relation

$$
\begin{aligned}
\det \mathcal{L}(\lambda) &= (1 + q_\lambda(x,x))\left(q_\lambda(y,y) + q_\lambda\left(\frac{\mu}{x},\frac{\mu}{x}\right) + \sigma\right) - q_\lambda(x,y)^2 \\
(5.2)\qquad &= \sigma + \sum_{s=0}^{r} \frac{\tilde{f}_s}{\lambda - \alpha_s} + \frac{P_s}{(\lambda - \alpha_s)^2} + \sum_{i=0}^{n} \frac{\mu_i^2}{(\lambda - a_i)^2},
\end{aligned}
$$

we get the integrals

$$
(5.3)\qquad \tilde{f}_s = \sum_{i \in I_s}\left(y_i^2 + \sigma x_i^2 + \frac{\mu_i^2}{x_i^2} + \sum_{j \notin I_s}\frac{P_{ij}}{a_i - a_j}\right),
$$

$$
(5.4)\qquad P_s = \sum_{i,j \in I_s, i<j} P_{ij}, \qquad k_s = |I_s| \geq 2,
$$

where P_{ij} are given by

$$
P_{ij} = (y_i x_j - x_i y_j)^2 + \frac{\mu_i^2 x_j^2}{x_i^2} + \frac{\mu_j^2 x_i^2}{x_j^2},
$$

while for $|I_s| = 1$ we set $P_s \equiv 0$. Whence, we have ρ nontrivial integrals among P_s, where ρ is the number of sets I_s for which $k_s = |I_s| \geq 2$.

In terms of integrals $\tilde{f}_s$, the Hamiltonian of the system can be express as

$$
H = \frac{1}{2}\sum_{s=0}^{r}\tilde{f}_s.
$$

THEOREM 5.1. *Apart of the integrals arising from the Lax representation, the rational functions*

$$
(5.5)\qquad P_{s,ij} := P_{ij}, \qquad i,j \in I_s
$$

are integrals of the Jacobi-Rosochatius problem (4.10). *The functions* $\tilde{f}_s, P_s$ *are central functions within the set of integrals* $\mathcal{F} = \{\tilde{f}_s, P_{s,ij}\}$:

$$
\begin{aligned}
&\{\tilde{f}_{s_1}, \tilde{f}_{s_2}\}_D = 0, &&\{\tilde{f}_{s_1}, P_{s_2}\}_D = 0, &&\{P_{s_1}, P_{s_2}\}_D = 0, \\
&\{\tilde{f}_{s_1}, P_{s_2,ij}\}_D = 0, &&\{P_{s_1}, P_{s_2,ij}\}_D = 0.
\end{aligned}
$$

Also, the functions $P_{s_1, i_1 j_1}$ and $P_{s_2, i_2 j_2}$ mutually commute for distinct s_1 and s_2,

$$(5.6) \qquad \{P_{s_1, i_1 j_1}, P_{s_2, i_2 j_2}\}_D = 0.$$

PROOF. The theorem can be verified by a straightforward calculations. Instead, we consider the Jacobi-Rosochatius problems on an one-parametric family of deformed, non-symmetric ellipsoids

$$E_\epsilon^n : \quad \langle A_\epsilon^{-1} x, x \rangle = 1, \quad A_\epsilon = \mathrm{diag}(a_0^\epsilon, \dots, a_n^\epsilon), \quad a_i^\epsilon \neq a_j^\epsilon, \quad i \neq j,$$

where

$$\lim_{\epsilon \to 0} a_i^\epsilon = a_i,$$

and a_i^ϵ are smooth functions defined on some interval $(-\Delta, \Delta)$.

Let $\{\cdot, \cdot\}_D^\epsilon$ be the associated Dirac-Poisson bracket (1.6) and let f_i^ϵ be the corresponding integrals (4.12),

$$(5.7) \qquad \{f_i^\epsilon, f_j^\epsilon\}_D^\epsilon = 0.$$

Define

$$\tilde{f}_s^\epsilon = \sum_{i \in I_s} f_i^\epsilon = \sum_{i \in I_s} \left(y_i^2 + \sigma x_i^2 + \frac{\mu_i^2}{x_i^2} + \sum_{j \notin I_s} \frac{P_{ij}}{a_i^\epsilon - a_j^\epsilon} \right), \quad s = 0, \dots, r,$$

$$P_{s,ij}^\epsilon = (a_i^\epsilon - a_j^\epsilon) f_i^\epsilon, \qquad i, j \in I_s.$$

From (5.7) we get $\{\tilde{f}_{s_1}^\epsilon, \tilde{f}_{s_2}^\epsilon\}_D^\epsilon = 0$ for all ϵ. Since $\lim_{\epsilon \to 0} \tilde{f}_s^\epsilon = \tilde{f}_s$, by taking the limit we obtain

$$\{\tilde{f}_{s_1}, \tilde{f}_{s_2}\}_D = 0.$$

On the other hand, $\lim_{\epsilon \to 0} P_{s,ij}^\epsilon$ can be singular and depends on the deformation. Suppose that

$$\lim_{\epsilon \to 0} \frac{a_i^\epsilon - a_j^\epsilon}{a_i^\epsilon - a_l^\epsilon} = 0, \qquad l \neq i, j.$$

Then $\lim_{\epsilon \to 0} P_{s,ij}^\epsilon = P_{s,ij}$ and from $\{\tilde{f}_{s_1}^\epsilon, P_{s_2,ij}^\epsilon\}_D^\epsilon = 0$ for all ϵ, we get

$$\{\tilde{f}_{s_1}, P_{s_2,ij}\}_D = 0.$$

Therefore

$$\{\tilde{f}_{s_1}, P_{s_2}\}_D = \sum_{i,j \in I_{s_2}, i < j} \{\tilde{f}_{s_1}, P_{s_2,ij}\}_D = 0$$

and

$$\{P_{s,ij}, H\}_D = \frac{1}{2} \sum_{s=0}^{r} \{P_{s,ij}, \tilde{f}_s\}_D = 0,$$

that is $P_{s,ij}$ are integrals of the system. Similarly, for $s_1 \neq s_2$, $i_1, j_1 \in I_{s_1}$, $i_2, j_2 \in I_{s_2}$, we can always find a perturbation A_ϵ such that

$$\lim_{\epsilon \to 0} \frac{a_{i_d}^\epsilon - a_{j_d}^\epsilon}{a_{i_d}^\epsilon - a_{l_d}^\epsilon} = 0, \qquad l_d \neq i_d, j_d, \qquad d = 1, 2.$$

Therefore, $\lim_{\epsilon \to 0} P^\epsilon_{s_d, i_d j_d} = P_{s_d, i_d j_d}$, $d = 1, 2$, and we have

$$\{P_{s_1, i_1 j_1}, P_{s_2, i_2 j_2}\}_D = 0,$$
$$\{P_{s_1}, P_{s_2, i_2, j_2}\} = \sum_{i_1, j_1 \in I_{s_1}, i_1 < j_1} \{P_{s_1, i_1 j_1}, P_{s_2, i_2 j_2}\}_D = 0,$$
$$\{P_{s_1}, P_{s_2}\} = \sum_{i_2, j_2 \in I_{s_2}, i_2 < j_2} \{P_{s_1}, P_{s_2, i_2 j_2}\}_D = 0.$$

It remains to prove $\{P_s, P_{s,ij}\}_D = 0$, $|I_s| \geq 3$. For simplicity, assume $s = 0, i = 0, j = 1$. Consider a deformation A_ϵ having the property

(5.8)
$$\lim_{\epsilon \to 0} \frac{a_0^\epsilon - a_1^\epsilon}{a_0^\epsilon - a_l^\epsilon} = 0, \qquad l \neq 0, 1,$$
$$\lim_{\epsilon \to 0} \frac{a_2^\epsilon - a_0^\epsilon}{a_2^\epsilon - a_l^\epsilon} = 0, \qquad l \neq 0, 1, 2$$
$$\lim_{\epsilon \to 0} \frac{a_2^\epsilon - a_0^\epsilon}{a_2^\epsilon - a_1^\epsilon} = 1.$$

For example, we can take $a_0^\epsilon = \alpha_0$, $a_1^\epsilon = \alpha_0 + \epsilon^3$, $a_2^\epsilon = \alpha_0 + \epsilon^3 + \epsilon^2$, $a_3^\epsilon = \alpha_0 + \epsilon^3 + \epsilon^2 + \epsilon$, ..., $a_{k_0} = \alpha_0 + \epsilon^3 + \epsilon^2 + (k_0 - 2)\epsilon$. Subsequently, we get

$$\lim_{\epsilon \to 0} P^\epsilon_{0,0,1} = \lim_{\epsilon \to 0} (a_0^\epsilon - a_1^\epsilon) f_0^\epsilon = P_{0,0,1},$$
$$\lim_{\epsilon \to 0} P^\epsilon_{0,2,0} = \lim_{\epsilon \to 0} (a_2^\epsilon - a_0^\epsilon) f_2^\epsilon = P_{0,2,0} + P_{0,2,1} = P_{0,0,2} + P_{0,1,2}$$

and, consequently,

(5.9) $$\{P_{0,0,1}, P_{0,1,2} + P_{0,0,2}\}_D = 0.$$

Next, we take $a_0^\epsilon = \alpha_0$, $a_1^\epsilon = \alpha_0 + \epsilon^2$, $a_2^\epsilon = \alpha_0 + \epsilon^2 + \epsilon$, $a_3^\epsilon = \alpha_0 + 2\epsilon^2 + \epsilon$, $a_4^\epsilon = \alpha_0 + 2\epsilon^2 + 2\epsilon$ ..., $a_{k_0} = \alpha_0 + 2\epsilon^2 + (k_0 - 2)\epsilon$. Then

$$\lim_{\epsilon \to 0} \frac{a_0^\epsilon - a_1^\epsilon}{a_0^\epsilon - a_l^\epsilon} = 0, \qquad l \neq 0, 1, \qquad \lim_{\epsilon \to 0} \frac{a_2^\epsilon - a_3^\epsilon}{a_2^\epsilon - a_l^\epsilon} = 0, \qquad l \neq 2, 3,$$

and $\lim_{\epsilon \to 0} P^\epsilon_{0,0,1} = P_{0,0,1}$, $\lim_{\epsilon \to 0} P^\epsilon_{0,2,3} = P_{0,2,3}$. Thus, we get

(5.10) $$\{P_{0,0,1}, P_{0,2,3}\}_D = 0.$$

Finally, repeating the arguments given for (5.9) and (5.10), we obtain the commuting relations

(5.11) $$\{P_{s,ij}, P_{s,ik} + P_{s,jk}\}_D = 0, \qquad \{P_{s,ij}, P_{s,kl}\}_D = 0, \qquad i, j \neq k, l,$$

implying

$$\{P_{s,ij}, P_s\}_D = \sum_{k, l \in I_s, k < l} \{P_{s,ij}, P_{s,kl}\}_D = \sum_{k \in I_s} \{P_{s,ij}, P_{s,ik}\}_D + \{P_{s,ij}, P_{s,jk}\}_D = 0.$$

$\square$

In particular, for the case of the Jacobi problem (where we set $\mu_0 = \cdots = \mu_n = 0$), or for the case of a geodesic flow on a symmetric ellipsoid ($\mu_0 = \cdots = \mu_n = \sigma = 0$), the system is invariant with respect to the $SO(k_0) \times \cdots \times SO(k_r)$ action and the integrals (5.5) reduce to the squares of the Noether integrals

$$(5.12) \qquad \Phi_{s,ij} = y_i x_j - x_i y_j, \qquad i < j, \qquad i,j \in I_s,$$

while the central functions P_s reduce to the invariants

$$(5.13) \qquad \Phi_s^2 = \sum_{i,j \in I_s, i<j} \Phi_{s,ij}^2.$$

The detail study of the geodesic flow is given in [**10**] (the case $n = 3$ can be found in [**11**]). It is proved that among the central functions $\tilde{f}_s, P_s$, with $\mu_0 = \cdots = \mu_n = \sigma = 0$, there are $r + \rho_0$ independent ones, while that among $\tilde{f}_s$ and the Noether integrals (5.12) there are $2n - r - \rho$ independent ones. Therefore, for a sufficiently small parameters μ_i and σ, we get that the dimensions of linear spaces

$$F(x,y) = \langle X_{\tilde{f}_s}(x,y), X_{P_{s,ij}}(x,y) \,|\, s = 0, 1, \ldots, r,\, i,j \in I_s \rangle,$$
$$K(x,y) = \langle X_{\tilde{f}_s}(x,y), X_{P_s}(x,y) \,|\, s = 0, 1, \ldots, r \rangle$$

are at least $2n - r - \rho$ and $r + \rho$, respectively, at a generic point $(x,y) \in T^*E^n$. Here X_f denotes the Hamiltonian vector field with respect to the Dirac-Poisson bracket (1.6). According to (5.2) we have the relation

$$(5.14) \qquad \sum_{s=0}^{r} \frac{\tilde{f}_s}{\alpha_s} = \sigma + \sum_{s=0}^{r} \frac{P_s}{\alpha_s^2} + \sum_{i=0}^{n} \frac{\mu_i^2}{a_i^2},$$

so $\dim K(x,y) \leq r + \rho$.

Since all object are rational functions, $\dim F(x,y) \geq 2n - r - \rho$ and $\dim K(x,y) = r + \rho$, for a generic values of μ_i, σ, $(x,y) \in T^*E^n$. As a result, we conclude that $\mathcal{F} = \{\tilde{f}_s, P_{s,ij}\}$ is a complete set of integrals (a generic dimension of $F(x,y)$ equals $2n - r - \rho$ and $\dim F(x,y) + \dim K(x,y) = \dim T^*E^n$) and we can apply the Nekhoroshev-Mishchenko-Fomenko theorem on noncommutative integrability (see [**38, 35, 3**]).

THEOREM 5.2. *The Jacobi-Rosochatius problem (4.10) (the Jacobi problem (1.3)) on a symmetric ellipsoid (1.1), (5.1) is completely integrable in a non-commutative sense by means of integrals (5.3) and (5.5) (where we set $\mu_i = 0$). Generic trajectories take place over $r + \rho$-dimensional invariant isotropic tori, spanned by the Hamiltonian vector fields $X_{\tilde{f}_s}, X_{P_s}$.*

In general, noncommutative integrability implies the usual Liouville integrability by means of smooth commuting integrals [**6**]. Here, the integrals can be chosen to be linear functions of non-commuting integrals.

THEOREM 5.3. *The Jacobi-Rosochatius problem (4.10) on a symmetric ellipsoid (1.1), (5.1) is Liouville integrable by means of integrals (5.3) and*

$$(5.15) \qquad L_{s,k} = \sum_{i,j \in I_{s,k}, i<j} P_{s,ij}, \qquad k = 1, \ldots, k_s - 1,\, s = 0, \ldots, r,$$

where

$$I_{0,k} = \{0, \ldots, k\},$$
$$I_{1,k} = \{k_0, \ldots, k_0 + k\},$$
$$\ldots$$
$$I_{r,k} = \{k_0 + \cdots + k_{r-1}, \ldots, k_0 + \cdots + k_{r-1} + k\}.$$

PROOF. According to (5.6), we have $\{L_{s_1,k_1}, L_{s_2,k_2}\}_D = 0$ for $s_1 \neq s_2$, while from (5.11) we get

$$\{P_{s,ij}, L_{s,k}\}_D = 0, \qquad i, j \in I_{s,k},$$

and, in particular,

$$\{L_{s,k_1}, L_{s,k_2}\}_D = 0$$

(note that $P_s = L_{s,k_s-1}$).

It is clear that the integrals (5.15) are mutually independent and their total number is

$$(k_0 - 1) + \cdots + (k_r - 1) = (n + 1) - (r + 1) = n - r.$$

On the other hand, from the completeness of $\mathcal{F} = \{\tilde{f}_s, P_{s,ij}\}$, r functions among $\tilde{f}_s$ are independent from the integrals $P_{s,ij}$ (we have the relation (5.14) among them). Therefore, the set of integrals $\{\tilde{f}_s, L_{s,k}\}$ is a complete commutative set on T^*E^n. $\qquad \square$

A choice of commuting integrals is not unique and (5.15) is motivated by a *chain of subalgebras method* in the construction of commutative functions on Lie algebras (e.g., see Thimm [43]). Another complete families of commuting integrals can be obtained, for example, by using separable variables related to the degeneration of the elliptic coordinates (see [7, 26]).

6. Billiards inside ellipsoids

6.1. Billiards: continuous and discrete description. Let (Q, g) be a n–dimensional Riemannian manifold and let $D \subset Q$ be a domain with a (smooth) boundary Γ. Let $\pi : T^*Q \to Q$ be a natural projection and let g^{-1} be the contravariant metric on the cotangent bundle. Consider the *reflection mapping* $r : \pi^{-1}\Gamma \to \pi^{-1}\Gamma$, $\mathbf{y}_- \mapsto \mathbf{y}_+$, which associates the covector $\mathbf{y}_+ \in T^*_{\mathbf{x}}Q$, $\mathbf{x} \in \Gamma$ to a covector $\mathbf{y}_- \in T^*_{\mathbf{x}}Q$ such that the following conditions hold:

$$|\mathbf{y}_+| = |\mathbf{y}_-|, \qquad \mathbf{y}_+ - \mathbf{y}_- \perp \Gamma.$$

A *billiard* in D is a dynamical system with the phase space T^*D whose trajectories are geodesics given by the Hamiltonian equations with the Hamiltonian $H(\mathbf{x}, \mathbf{y}) = \frac{1}{2}g^{-1}(\mathbf{y}, \mathbf{y})$, reflected at points $\mathbf{x} \in \Gamma$ according to the billiard law: $r(\mathbf{y}_-) = \mathbf{y}_+$. Here $\mathbf{y}_-$ and $\mathbf{y}_+$ denote the momenta before and after the reflection. If some potential force field $V(\mathbf{x})$ is added than the system is described with the same reflection law and Hamiltonian equations with the Hamiltonian $H(\mathbf{x}, \mathbf{y}) = \frac{1}{2}g^{-1}(\mathbf{y}, \mathbf{y}) + V(\mathbf{x})$.

A function $f : T^*Q \to \mathbb{R}$ is an *integral* of the billiard system if it commutes with the Hamiltonian ($\{f, H\} = 0$) and does not change under the reflection ($f(\mathbf{x}, \mathbf{y}) =$

$f(\mathbf{x}, r(\mathbf{y}))$, $\mathbf{x} \in \Gamma$). The billiard is *completely integrable in the sense of Birkhoff* if it has n independent integrals polynomial in the momenta, which are in involution (see [**29**]).

It is well known that the billiard system within an ellipsoid $E^{n-1} \subset \mathbb{R}^n$ under the influence of an arbitrary potential separable in elliptic coordinates is completely integrable in the sense of Birkhoff [**29, 13, 14**]. Moreover, the manifold $T^*D/r \smallsetminus \Sigma$ carries well defined symplectic structure, such that the billiard flow is the usual Hamiltonian flow with n commuting smooth integrals (Σ is the codimension two submanifold – the cotangent bundle of the ellipsoid, see Lazutkin [**32**]). Whence we can use the Arnold-Liouville theorem in the description of the system. Alternatively, we can consider the billiard as a discrete integrable system with the *billiard mapping* $\phi : (\mathbf{x}_k, \mathbf{y}_k) \mapsto (\mathbf{x}_{k+1}, \mathbf{y}_{k+1})$, where $\mathbf{x}_k$ is a sequence of the points of impact and $\mathbf{y}_k$ is the corresponding sequence of outgoing momenta [**46**]. However, it is not a simple task to explicitly describe the billiard map in the Descartes coordinates, as it the case when the trajectories between the impacts are the straight lines.

The first result in this direction is performed by Fedorov [**21**], who calculated the billiard map and found the Lax representation for the billiard system under the influence of the elastic force [**49**]. As a slight modification, in this section we consider the billiard system under the additional influence of the Rosochatius potential. In the derivation of the billiard mapping we use a discrete version of the reduction given in Section 4.

6.2. Harmonic oscillator and ellipsoidal billiards. Consider the Jacobi problem (4.3) on the $2n + 1$-dimensional ellipsoid E^{2n+1}. When parameter a_0 tends to zero, the Jacobi flow transforms to the billiard problem within real $2n - 1$-dimensional ellipsoid in $\mathbb{C}^n$

$$(6.1) \qquad E^{2n-1} = \{\mathbf{z} \in \mathbb{C}^n \,|\, \langle a^{-1}\mathbf{z}, \bar{\mathbf{z}} \rangle = 1\},$$

where the motion between the impacts is influenced by the elastic force $-\sigma z$ (harmonic oscillations constrained inside the ellipsoid (6.1)):

$$(6.2) \qquad \dot{\mathbf{z}} = \mathbf{p}, \qquad \dot{\mathbf{p}} = -\sigma \mathbf{z}.$$

Here we denoted $a = \mathrm{diag}(a_1, \ldots, a_n)$, $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{p} = (p_1, \ldots, p_n)$.

If $\sigma \leq 0$, then all trajectories have reflections from the boundary E^{2n-1}, while for $\sigma > 0$, the initial conditions $(\mathbf{z}_0, \mathbf{p}_0)$ determining the energy $h = H(\mathbf{z}_0, \mathbf{p}_0)$ should satisfy

$$h + \frac{\sigma}{2} \langle \mathbf{z}, \bar{\mathbf{z}} \rangle > \epsilon > 0, \qquad \mathbf{z} \in E^{2n-1}.$$

Consider the integral

$$(6.3) \qquad J = \langle A^{-1}p, \bar{p} \rangle \langle A^{-2}z, \bar{z} \rangle - \sigma \langle A^{-2}z, \bar{z} \rangle$$

of the Jacobi problem (4.3). For distinct a_i, it is equal to the sum $-\sum_i a_i^{-2} f_i$. Also, note that H/J for $\sigma = 0$ equals to the square of the Joachimsthal integral of the geodesic flow [**36**].

In the limiting process, the integral (6.3) multiplied by a_0 becomes

$$(6.4) \qquad \lim_{a_0 \to 0} a_0 J = \frac{1}{4}((\langle a^{-1}\mathbf{z}, \bar{\mathbf{p}}\rangle + \langle a^{-1}\bar{\mathbf{z}}, \mathbf{p}\rangle)^2 + (1 - \langle a^{-1}\mathbf{z}, \bar{\mathbf{z}}\rangle)(\langle a^{-1}\mathbf{p}, \bar{\mathbf{p}}\rangle - \sigma).$$

Let $\mathbf{z}_k$, $k \in \mathbb{Z}$, be the set of impact points. By $\mathbf{p}_k$ we denote the outgoing velocity at $\mathbf{z}_k$. From (6.4) we get the integral

$$J_k = \langle a^{-1}\mathbf{z}_k, \bar{\mathbf{p}}_k\rangle + \langle a^{-1}\bar{\mathbf{z}}_k, \mathbf{p}_k\rangle.$$

of the billiard mapping within ellipsoid (6.1)

$$(6.5) \qquad \phi(\mathbf{z}_k, \mathbf{p}_k) = (\mathbf{z}_{k+1}, \mathbf{p}_{k+1}).$$

With the above notation, the mapping (6.5) given in [**21**] reads

$$\mathbf{z}_{k+1} = -\frac{1}{\nu_k}[(\sigma - \langle \mathbf{p}_k, a^{-1}\bar{\mathbf{p}}_k\rangle)\mathbf{z}_k + J_k\mathbf{p}_k],$$

$$\mathbf{p}_{k+1} = -\frac{1}{\nu_k}[(\sigma - \langle \mathbf{p}_k, a^{-1}\bar{\mathbf{p}}_k\rangle)(\mathbf{p}_k + \pi_k a^{-1}\mathbf{z}_k) + J_k(\pi_k a^{-1}\mathbf{p}_k - \sigma\mathbf{z}_k)],$$

where

$$\nu_k = \sqrt{\sigma J_k^2 + (\sigma - \langle \mathbf{p}_k, a^{-1}\bar{\mathbf{p}}_k\rangle)^2}, \quad \pi_k = J_k/\langle \mathbf{z}_{k+1}, a^{-2}\bar{\mathbf{z}}_{k+1}\rangle.$$

6.3. The Jacobi-Rosochatius billiard. The harmonic oscillations with impacts at the boundary (6.1) have integrals

$$(6.6) \qquad h_j(\mathbf{z}_k, \mathbf{p}_k) = \frac{i}{2}(z_{k,j}\bar{p}_{k,j} - p_{k,j}\bar{z}_{k,j}), \quad j = 1, \ldots, n,$$

and we can perform a discrete analogue of the reduction described in Section 4.

Let us fix the values of the integrals

$$h_j = \mu_j, \quad j = 1, \ldots, n,$$

where some of μ_j can be equal to zero. Introduce the coordinate change

$$(6.7) \qquad z_{k,j} = x_{k,j}e^{i\varphi_{k,j}}, \quad p_{k,j} = y_{k,j}e^{i\varphi_{k,j}} + i\frac{\mu_j}{x_{k,j}}e^{i\varphi_{k,j}} \quad \text{for} \quad \mu_j \neq 0$$

and consider the restrictions of $z_{k,j}, p_{k,j}$ to $\mathbb{R}$,
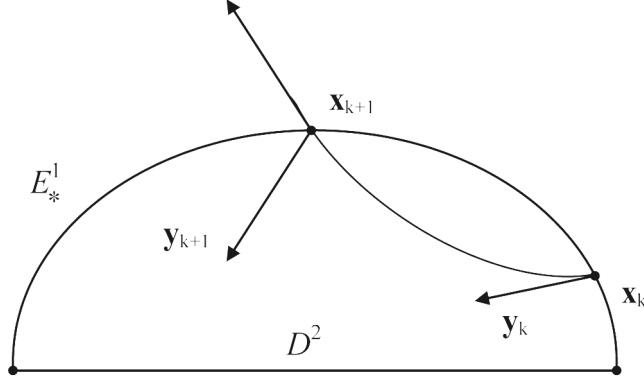
$$z_{k,j} = x_{k,j} \in \mathbb{R}, \quad p_{k,j} = y_{k,j} \in \mathbb{R} \quad \text{for} \quad \mu_j = 0.$$

The mapping (6.5), induces the mapping

$$(6.8) \qquad \Phi(\mathbf{x}_k, \mathbf{y}_k) = (\mathbf{x}_{k+1}, \mathbf{y}_{k+1}), \qquad \mathbf{x}_k, \mathbf{x}_{k+1} \in E_*^{n-1}$$

where E_*^{n-1} is the ellipsoidal component of the boundary of the domain

$$D^n = \{\mathbf{x} \in \mathbb{R}^n \mid \langle \mathbf{x}, a^{-1}\mathbf{x}\rangle \leq 1, \ x_j \geq 0 \text{ for } \mu_j \neq 0, j = 1, \ldots, n\}.$$

FIGURE 1. Billiard domain for $n = 2$, $\mu_1 = 0$, $\mu_2 \neq 0$.

LEMMA 6.1.

$$x_{k+1,j} = \frac{1}{\nu_k}\sqrt{\left(J_k y_{k,j} + K_k x_{k,j}\right)^2 + \frac{\mu_j^2}{x_{k,j}^2}}, \qquad \mu_j \neq 0,$$

$$x_{k+1,j} = -\frac{1}{\nu_k}\left[K_k x_{k,j} + J_k y_{k,j}\right], \qquad\qquad \mu_j = 0,$$

$$y_{k+1,j} = -\frac{e^{-i\delta_{k,j}}}{\nu_k}\left[K_k\left(\frac{\pi_k}{a_j}x_{k,j} + y_{k,j} + i\frac{\mu_j}{x_{k,j}}\right)\right]$$
$$-\frac{J_k e^{-i\delta_{k,j}}}{\nu_k}\left[\frac{\pi_k}{a_j}y_{k,j} - \sigma x_{k,j} + i\frac{\pi_k \mu_j}{a_j x_{k,j}}\right] - i\frac{\mu_j}{x_{k+1,j}},$$

where

$$J_k = 2\langle \mathbf{x}_k, a^{-1}\mathbf{y}_k\rangle,$$

$$K_k = \sigma - \langle \mathbf{y}_k, a^{-1}\mathbf{y}_k\rangle - \left\langle \frac{\mu}{\mathbf{x}_k}, a^{-1}\frac{\mu}{\mathbf{x}_k}\right\rangle,$$

$$\nu_k = \sqrt{\sigma J_k^2 + K_k^2},$$

$$\pi_k = J_k/\langle \mathbf{x}_{k+1}, a^{-2}\mathbf{x}_{k+1}\rangle,$$

$$\delta_{k,j} = Arg\left[-K_k x_{k,j} - J_k y_{k,j} - iJ_k\frac{\mu_j}{x_{k,j}}\right].$$

This is a billiard mapping of the billiard system within a domain D^n, where the motion between the impacts is described by the Hamiltonian function

$$H(\mathbf{x}, \mathbf{y}) = \frac{1}{2}\langle \mathbf{y}, \mathbf{y}\rangle + \frac{\sigma}{2}\langle \mathbf{x}, \mathbf{x}\rangle + \frac{1}{2}\sum_{j=1}^{n}\frac{\mu_k^2}{x_j^2}$$

that is,

$$(6.9) \qquad\qquad \dot{\mathbf{x}} = \mathbf{y}, \qquad \dot{\mathbf{x}} = -\sigma\mathbf{x} + \frac{\mu^2}{\mathbf{x}^3},$$

where $\mu^2/\mathbf{x}^3 = (\mu_1^2/x_1^3, \ldots, \mu_n^2/x_n^3)$.

The trajectories $\mathbf{x}(t)$ of (6.9) are projections of trajectories $\mathbf{z}(t)$ of the harmonic oscillator (6.2) (conics or degenerate conics) by the reduction (6.7). As above, if $\sigma \leq 0$, then all trajectories will have reflections from the boundary, while for $\sigma > 0$, the initial conditions $(\mathbf{x}_0, \mathbf{p}_0)$ determining the energy $h = H(\mathbf{x}_0, \mathbf{y}_0)$ should satisfy

$$h + \frac{\sigma}{2}\langle \mathbf{x}, \mathbf{x}\rangle > \epsilon > 0, \qquad \mathbf{x} \in E_*^{n-1}.$$

We refer to (6.8) as the *Jacobi-Rosochatius billiard mapping*. After a straightforward modification of Fedorov's Lax representation [**21**] by applying a discrete reduction, we obtain the following statement.

THEOREM 6.1. *Suppose that the eigenvalues a_i of the matrix a are distinct. Up to the action of the group generated by the reflections*

$$(6.10) \qquad (x_j, y_j) \mapsto (s_j x_j, s_j y_j), \quad s_j = \pm 1, \quad \mu_j = 0, \quad s_j = 1, \quad \mu_j \neq 0,$$

$j = 1, \ldots, n$, *the Jacobi-Rosochatius billiard map* (6.8) *is equivalent to the matrix equation*

$$(6.11) \qquad \mathcal{L}_{\mathbf{x}_{k+1}, \mathbf{y}_{k+1}}(\lambda) = \mathcal{A}_{\mathbf{x}_k, \mathbf{y}_k}(\lambda)\mathcal{L}_{\mathbf{x}_k, \mathbf{y}_k}(\lambda)\mathcal{A}_{\mathbf{x}_k, \mathbf{y}_k}^{-1}(\lambda)$$

with 2×2 matrices depending on the parameter λ

$$(6.12) \qquad \mathcal{L}_{\mathbf{x}_k, \mathbf{y}_k}(\lambda) = \begin{pmatrix} Q_\lambda(\mathbf{x}_k, \mathbf{y}_k) & Q_\lambda(\mathbf{y}_k, \mathbf{y}_k) + Q_\lambda(\frac{\mu}{\mathbf{x}_k}, \frac{\mu}{\mathbf{x}_k}) + \sigma \\ -1 - Q_\lambda(\mathbf{x}_k, \mathbf{x}_k) & -Q_\lambda(\mathbf{y}_k, \mathbf{x}_k) \end{pmatrix},$$

$$\mathcal{A}_{\mathbf{x}_k, \mathbf{y}_k}(\lambda) = \begin{pmatrix} K_k\lambda + J_k\pi_k & \sigma J_k\lambda - K_k\pi_k \\ -J_k\lambda & K_k\lambda \end{pmatrix},$$

where $Q_\lambda(\cdot, \cdot) = \langle a_\lambda^{-1}\cdot, \cdot\rangle$, $a_\lambda = \mathrm{diag}(\lambda - a_1, \ldots, a_n - \lambda)$.

REMARK 6.1. Note that the billiard can be seen as a limit of the system (4.10) as a_0 tends to zero and $\mu_0 = 0$. Contrary, as the value of the integral $J_k = 2\langle \mathbf{x}_k, \mathbf{y}_k\rangle$ tends to zero, the impact points $\mathbf{x}_k$ approximate the trajectories of the Jacobi-Rosochatius problem on E_*^{n-1}. As in the case of the Jacobi-Rosochatius system (4.10) on a symmetric ellipsoid, the Jacobi-Rosochatius billiard map (6.8) for a symmetric ellipsoid (see the equation (6.22) given below) is an example of a discrete system integrable in a noncommutative sense.

6.4. The Chasles and Poncelet theorems. In what follows we shall give a geometric interpretation of the integrability.

Let us recall on a well known variant of the Chasles theorem for the billiard system within ellipsoid

$$(6.13) \qquad\qquad E^{n-1} = \{\mathbf{x} \in \mathbb{R}^n \,|\, \langle \mathbf{x}, a^{-1}\mathbf{x}\rangle = 1\}$$

without external forces, e.g., see [**29, 46, 16**]. Assume that the eigenvalues of the matrix a are different. Let $\mathbf{x}_k$ be a generic sequence of impact points. Then

the sequence of lines $\mathbf{x}_k \mathbf{x}_{k+1}$ is simultaneously tangent to the same set of quadrics $\mathcal{Q}_{\eta_1}, \ldots, \mathcal{Q}_{\eta_{n-1}}$ from the confocal family

$$(6.14) \qquad \mathcal{Q}_\lambda: \qquad \sum_{i=1}^{n} \frac{x_i^2}{a_i - \lambda} = 1.$$

Namely, the set of lines $p = p(\mathbf{x}, \mathbf{y}) = \{\mathbf{x} + s\mathbf{y} \mid s \in \mathbb{R}\}$ that are tangent to the quadric $\mathcal{Q}_\eta$ from the confocal family (6.14) are given by the equation [36]

$$(6.15) \qquad \Phi_{\mathbf{x}, \mathbf{y}}(\eta) = (Q_\eta(\mathbf{x}, \mathbf{x}) + 1) Q_\eta(\mathbf{y}, \mathbf{y}) - Q_\eta^2(\mathbf{x}, \mathbf{y}) = 0.$$

On the other hand, from (6.11) where we set $\sigma = 0$ and $\mu_i = 0$, $\Phi_{\mathbf{x}_k, \mathbf{y}_k}(\lambda) = \det \mathcal{L}_{\mathbf{x}_k, \mathbf{y}_k}(\lambda)$ is an integral of the system. Therefore, if η is a zero of $\Phi_{\mathbf{x}_k, \mathbf{y}_k}(\lambda)$, then the lines $p_k = p(\mathbf{x}_k, \mathbf{y}_k) = \mathbf{x}_k \mathbf{x}_{k+1}$, $k \in \mathbb{Z}$ are simultaneously tangent to $\mathcal{Q}_\eta$. For a generic trajectory $(\mathbf{x}_k, \mathbf{y}_k)$, we have $n - 1$ distinct solutions $\eta_1, \ldots, \eta_{n-1}$ of the equation $\Phi_{\mathbf{x}_k, \mathbf{y}_k}(\lambda) = 0$.

Moreover, suppose that the trajectory $\mathbf{x}_k$ is periodic. Then any billiard trajectory which shares the same caustic quadrics is also periodic, with the same period (*the Poncelet theorem* in $\mathbb{R}^n$ [**39, 9, 15, 16**]).

An analytical condition on caustics $\mathcal{Q}_{\eta_1}, \ldots, \mathcal{Q}_{\eta_{n-1}}$ for periodic billiard trajectories is derived by Dragović and Radnović, generalizing classical Cayley's condition for $n = 2$ [**15, 16**]. The geometry of the lines common to the confocal quadrics is further studied in [**30, 16**], while Chasles's-type theorems for several natural mechanical systems are given in [**33, 19, 22**].

Let d be the number of indexes i for which $\mu_i \neq 0$.

THEOREM 6.2. *Assume that the eigenvalues of the matrix a are different. Let* $(\mathbf{x}_k, \mathbf{y}_k)$ *be a generic trajectory of the Jacobi-Rosochatius billiard map (6.8).*

(i) Assume $\sigma \neq 0$ (respectively, $\sigma = 0$). The trajectories

$$(6.16) \qquad l_k: \qquad \mathbf{x}_k(t), \qquad t \in \mathbb{R}$$

of the Jacobi-Rosochatius system (6.9) with initial conditions $(\mathbf{x}_k(0), \mathbf{y}_k(0)) = (\mathbf{x}_k, \mathbf{y}_k)$ are simultaneously tangent to the quadrics

$$(6.17) \qquad \mathcal{Q}_{\eta_1}, \ldots, \mathcal{Q}_{\eta_{n+d}} \qquad (\text{respectively}, \ \mathcal{Q}_{\eta_1}, \ldots, \mathcal{Q}_{\eta_{n-1+d}})$$

from the confocal family (6.14), where $\eta_1, \ldots, \eta_{n+d}$ (respectively, $\eta_1, \ldots, \eta_{n-1+d}$) are solutions of the equation $\det \mathcal{L}_{\mathbf{x}_k, \mathbf{y}_k}(\lambda) = 0$.

(ii) Suppose that the trajectory $(\mathbf{x}_k, \mathbf{y}_k)$ is periodic. Then any billiard trajectory which shares the same caustic quadrics (6.17) is also periodic with the same period.

PROOF. (i) Firstly, we take $\sigma = 0$. For a sake of simplicity, assume $\mu_1, \ldots, \mu_d \neq 0$, $\mu_{d+1} = \cdots = \mu_n = 0$.

Consider the billiard system without external forces within a symmetric ellipsoid E^{n+d-1} in

$$\mathbb{R}^{n+d} \cong \mathbb{C}^d(z_1, \ldots, z_d) \times \mathbb{R}^{n-d}(x_{d+1}, \ldots, x_d),$$

where we take $a = (a_1, a_1, \ldots, a_d, a_d, a_{d+1}, a_{d+2}, \ldots, a_n)$, $a_i \neq a_j$, $i \neq j$. Let

$$(6.18) \quad (\tilde{\mathbf{x}}_k, \tilde{\mathbf{y}}_k) = (z_{1,k}, \ldots, z_{d,k}, x_{d+1,k}, \ldots, x_{n,k}, p_{1,k}, \ldots, p_{d,k}, y_{d+1,k}, \ldots, y_{n,k})$$

be a generic billiard trajectory with the values of the integrals

$$(6.19) \qquad h_j = \frac{i}{2}(z_{k,j}\bar{p}_{k,j} - p_{k,j}\bar{z}_{k,j}) = \mu_j, \quad j = 1, \dots, d.$$

From Lemma (6.2) below, the lines $\tilde{p}_k = \tilde{\mathbf{x}}_k \tilde{\mathbf{x}}_{k+1} \subset \mathbb{C}^d \times \mathbb{R}^{n-d}$ determined by the billiard trajectory (6.18) are tangent to the $N = n + d - 1$ quadrics

$$(6.20) \qquad \tilde{\mathcal{Q}}_{\eta_i} : \frac{z_1^2}{a_1 - \eta_l} + \cdots + \frac{z_d^2}{a_d - \eta_l} + \frac{x_{d+1}^2}{a_{d+1} - \eta_l} + \cdots + \frac{x_n^2}{a_n - \eta_l} = 1,$$

where η_l are zeros of the corresponding polynomial (6.23).

By applying the reduction (6.7) in the variables $z_1, \dots, z_d$ we get the billiard trajectory $(\mathbf{x}_k, \mathbf{y}_k)$ of the Rosochatius billiard map (6.8) with $\sigma = 0$, $\mu_1, \dots, \mu_d \neq 0$, $\mu_{d+1} = \cdots = \mu_n = 0$. At the same time, $\det \mathcal{L}_{\mathbf{x}_k, \mathbf{y}_k}(\eta_i) = 0$, $i = 1, \dots, n + d - 1$ and the lines $\tilde{p}_k$ project to the curves (6.16) tangent to the quadrics (6.17).

In the other direction, let $(\mathbf{x}_k, \mathbf{y}_k)$ be a trajectory of the Rosochatius billiard map (6.8) with $\sigma = 0$, $\mu_1, \dots, \mu_d \neq 0$, $\mu_{d+1} = \cdots = \mu_n = 0$. Then we can lift $(\mathbf{x}_k, \mathbf{y}_k)$ to the $SO(2)^d$-invariant family of billiard trajectories (6.18) of the billiard system without external forces within E^{n+d-1} satisfying (6.19).

The case $\sigma \neq 0, \mu_1 = \cdots = \mu_n = 0$ is proved by Fedorov [21]. For $\sigma \neq 0$, instead of the equation (6.15), which characterizes tangent lines to the quadric $\mathcal{Q}_\eta$, we use the equation

$$(6.21) \qquad \Phi^\sigma_{\mathbf{x}, \mathbf{y}}(\eta) = (Q_\eta(\mathbf{x}, \mathbf{x}) + 1)(Q_\eta(\mathbf{y}, \mathbf{y}) + \sigma) - Q_\eta^2(\mathbf{x}, \mathbf{y}) = 0.$$

A conic $l = \{\mathbf{x}(t) \mid t \in \mathbb{R}\}$ associated to a solution of the equation

$$\dot{\mathbf{x}}(t) = \mathbf{y}, \qquad \dot{\mathbf{y}}(t) = -\sigma \mathbf{x}$$

with the initial condition $(\mathbf{x}_0, \mathbf{y}_0)$ is tangent to $\mathcal{Q}_\eta$ if and only if $\Phi^\sigma_{\mathbf{x}, \mathbf{y}}(\eta) = 0$ (Proposition 1 in [21]). The statement can be applied for matrices a with multiple eigenvalues and repeating the arguments given above, item (i) follows.

(ii) Suppose that the trajectory $(\mathbf{x}_k, \mathbf{y}_k)$ with the energy $H = h$ is periodic. The Jacobi-Rosochatius billiard (6.8) can be considered as a usual integrable Hamiltonian system on $T^*D/r \smallsetminus \Sigma$ (we take only trajectories which are not tangent to the boundary E^{n-1}_*, that is $J_k = 2\langle \mathbf{x}_k, \mathbf{y}_k \rangle \neq 0$), see [32]. The associated continuous billiard trajectory $(\mathbf{x}(t), \mathbf{y}(t))$ is periodic as well.

The parameters of caustics $\eta_1, \dots, \eta_{n+d}$ (respectively, $\eta_1, \dots, \eta_{n+d-1}$) correspond to the unique values $c_1, \dots, c_n$, $c_1 + \cdots + c_n = 2h$ of the integrals (4.12) (where we set $i, j = 1, \dots, n$) determined from the equations

$$\det \mathcal{L}_{\mathbf{x}(t), \mathbf{y}(t)}(\eta_l) = \sigma + \sum_{i=1}^n \frac{c_i}{\eta_l - a_i} + \sum_{i=1}^n \frac{\mu_i^2}{(\eta_l - a_i)^2} = 0.$$

The curves (6.16) tangent to the caustics (6.17) are associated to billiard trajectories which belong to the Lagrangian tori that are components of the invariant level set

$$M_{c_1, \dots, c_n} \subset T^*D/r \smallsetminus \Sigma : \qquad f_1 = c_1, \dots, f_n = c_n.$$

According to the Arnold-Liuville theorem, if a trajectory on a regular Lagrangian torus is periodic, all trajectories on the same torus are also periodic with the same period. Since all components of $M_{c_1,\ldots,c_n}$ are related by the reflections (6.10), it is clear that if one Lagrangian torus is periodic, so are the others one. $\square$

We note that there is a natural generalization of the presented results to the ellipsoidal billiards on spheres and hyperbolic spaces [**29, 47, 14, 42**].

LEMMA 6.2. [The Chasles theorem for a symmetric ellipsoid] *Consider a billiard within a symmetric ellipsoid E^{n-1} defined by*

$$(6.22) \quad a_i = \alpha_1, \qquad i \in I_1 = \{1, \ldots, k_1\}, \ldots,$$
$$a_i = \alpha_r, \qquad i \in I_r = \{k_1 + \cdots + k_{r-1} + 1, \ldots, k_0 + \cdots + k_{r-1} + k_r\},$$

$\alpha_i \neq \alpha_j, i \neq j$, $k_1 + \cdots + k_r = n$. Let $\mathbf{x}_k$ be a generic sequence of impact points. Then the sequence of lines $\mathbf{x}_k\mathbf{x}_{k+1}$ is simultaneously tangent to the same set of quadrics $\mathcal{Q}_{\eta_1}, \ldots, \mathcal{Q}_{\eta_N}$ from the confocal family (6.14), where $N = \delta_1 + \cdots + \delta_r - 1$, $\delta_s = 2$ for $k_s = |I_s| \geq 2$, and $\delta_s = 1$ for $k_s = |I_s| = 1$.[2] Further, due to the $SO(k_1) \times \cdots \times SO(k_r)$ symmetry of the system, if $\mathbf{x}_k$ is a billiard trajectory, so is $R(\mathbf{x}_k)$, $R \in SO(k_1) \times \cdots \times SO(k_r)$, and the lines $R(\mathbf{x}_k\mathbf{x}_{k+1})$ are tangent to the same set of N quadrics.

PROOF. Note that the description (6.15) of the tangent lines to the quadric $\mathcal{Q}_\eta$ does hold for the matrices a with multiple eigenvalues (6.22). According to (5.2), the number of quadrics tangent to a generic line $p = p(\mathbf{x}, \mathbf{y})$ equals to the number of zeros of the polynomial

$$\Psi_{\mathbf{x},\mathbf{y}}(\lambda) = (\lambda - \alpha_1)^{\delta_1} \cdots (\lambda - \alpha_r)^{\delta_r} \det \mathcal{L}_{\mathbf{x},\mathbf{y}}(\lambda)$$
$$(6.23) \qquad = \sum_{s=1}^{r} \left((\lambda - \alpha_s)^{\delta_s - 1} \prod_{i \neq s} (\lambda - \alpha_i)^{\delta_i} \tilde{f}_s + \prod_{i \neq s} (\lambda - \alpha_i)^{\delta_i} P_s \right),$$

where $\tilde{f}_s$, P_s are given by (5.3), (5.4) (where we set $\sigma = 0$, $\mu_i = 0$). Therefore, a generic line $p = p(\mathbf{x}, \mathbf{y})$ is tangent to $N = \delta_1 + \cdots + \delta_r - 1$ quadrics from the family (6.14).

Consider a generic billiard trajectory $(\mathbf{x}_k, \mathbf{y}_k)$ within a symmetric ellipsoid (6.13), (6.22). As above, if the line $p_k = p(\mathbf{x}_k, \mathbf{y}_k) = \mathbf{x}_k\mathbf{x}_{k+1}$ is tangent to the quadric $\mathcal{Q}_\eta$ defined by the equation (6.14), i.e, $\Psi_{\mathbf{x}_k,\mathbf{y}_k}(\eta) = 0$, then the lines p_k, $k \in \mathbb{Z}$ are also tangent to $\mathcal{Q}_\eta$. $\square$

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[2]Note that $N = r + \rho - 1$, where $r + \rho$ is the dimension of invariant isotropic tori of the corresponding geodesic flow on the symmetric ellipsoid (1.1) in $\mathbb{R}^{n+1}$ with semi-axis $\sqrt{a_0} \approx 0$ and $\sqrt{a_1}, \ldots, \sqrt{a_n}$ given by (6.22) (Theorem 5.2).

7. Hierarchy of the Lax representations

By taking the limit $a_0 \to 0$ and assuming $\mu_0 = 0$, from Theorem 4.1 we can write down the Lax representation

$$\dot{\mathcal{L}}_{\mathbf{x},\mathbf{y}}(\lambda) = [\mathcal{L}_{\mathbf{x},\mathbf{y}}(\lambda), \mathcal{A}_{\mathbf{x},\mathbf{y}}(\lambda)],$$

$$\mathcal{L}_{\mathbf{x},\mathbf{y}}(\lambda) = \begin{pmatrix} Q_\lambda(\mathbf{x},\mathbf{y}) & Q_\lambda(\mathbf{y},\mathbf{y}) + Q_\lambda(\frac{\mu}{\mathbf{x}},\frac{\mu}{\mathbf{x}}) + \sigma \\ -1 - Q_\lambda(\mathbf{x},\mathbf{x}) & -Q_\lambda(\mathbf{y},\mathbf{x}) \end{pmatrix},$$

$$\mathcal{A}_{\mathbf{x},\mathbf{y}}(\lambda) = \begin{pmatrix} 0 & -\sigma \\ 1 & 0 \end{pmatrix}.$$

for the Jacobi-Rosochatius system on $\mathbb{R}^n$ (6.9), i.e, for the trajectories of the billiard system between the impacts. Here we used

$$\frac{\sigma - \langle A^{-1}y, y \rangle - \langle A^{-1}\frac{\mu}{x}, \frac{\mu}{x} \rangle}{\langle x, A^{-2}x \rangle} = \frac{\sigma - \langle a^{-1}\mathbf{y}, \mathbf{y} \rangle - \frac{a_0(1 - 2\langle \mathbf{x}, a^{-1}\mathbf{y} \rangle)^2}{4(1 - \langle \mathbf{x}, a^{-1}\mathbf{x} \rangle)} - \langle a^{-1}\frac{\mu}{\mathbf{x}}, \frac{\mu}{\mathbf{x}} \rangle}{\langle \mathbf{x}, a^{-2}\mathbf{x} \rangle + \frac{1}{a_0}(1 - \langle \mathbf{x}, a^{-1}\mathbf{x} \rangle)} \to 0,$$

as $a_0 \to 0$, where $x = (x_0, \mathbf{x})$ and $y = (y_0, \mathbf{y})$ satisfy the constraints (1.4).

This Lax representation, for $\mu_1 = \cdots = \mu_n$, is equivalent to the Lax representation for the harmonic oscillator given in [**17**], where, by induction, the Lax representations for natural mechanical systems with polynomial potentials separable in elliptic coordinates in $\mathbb{R}^n$ are given (see also [**2**]). In the same way, by using Theorem 4.1, we can give the 2×2-Lax representations for the separable potential perturbations of the Jacobi-Rosochatius system on E^n.

7.1. Separable potentials in $\mathbb{R}^{n+1}$. Recall, a potential $V(x)$ is separable in the elliptic coordinates $\lambda_0 < a_0 < \lambda_1 < a_1 < \cdots < \lambda_n < a_n$ defined by (1.7) if and only if it is a solution of the Bertrand-Darboux equations

$$(7.1) \quad (a_i - a_j)\frac{\partial^2 V}{\partial x_i \partial x_j} + \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right) \left(2V + \sum_{k=0}^{n+1} x_k \frac{\partial V}{\partial x_k} \right) = 0, \quad i \neq j$$

(Benenti [**4**], see also Marshall and Wojciechowski [**34**]). The solutions of (7.1) can be written in the form $V(x) = \sum_i x_i \frac{\partial \mathcal{V}}{\partial x_i}$, where $\mathcal{V}(x)$ are solutions of

$$(a_i - a_j)\frac{\partial^2 \mathcal{V}}{\partial x_i \partial x_j} = \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right) \left(\sum_k x_k \frac{\partial \mathcal{V}}{\partial x_k} \right), \quad i \neq j$$

(see [**51**]). Then, a complete set of commuting integrals is given by

$$(7.2) \quad f_i = y_i^2 + \sum_{j \neq i} \frac{(y_i x_j - y_j x_i)^2}{a_i - a_j} + 2F_i(x), \quad i = 0, 1, \ldots, n,$$

where $F_i(x) = x_i \frac{\partial \mathcal{V}}{\partial x_i}$ [**34, 51**].

Polynomial potentials are described in [**50, 51, 27**]. Basic homogeneous polynomial solutions $V^{(k)}$ of degree $2k$ of the equations (7.1) in elliptic coordinates

reads

$$V^{(k)}(\lambda_0,\ldots,\lambda_n) = -\sum_{j=0}^{n+1} \frac{\lambda_j^{k-1}\prod_i(\lambda_j - a_i)}{\prod_{i\neq j}(\lambda_j - \lambda_i)}$$

and $V^{(k)}, F_0^{(k)},\ldots,F_n^{(k)}$, $k \in \mathbb{N}$ satisfy the system of the recurrence relations

$$(7.3) \qquad F_i^{(k+1)} = a_i F_i^{(k)} - x_i^2 V^{(k)}, \quad F_i^{(1)} = x_i^2, \quad V^{(k)}(x) = F_0^{(k)} + \cdots + F_n^{(k)}$$

(we use the notation given by Zaitsev [51]). For example,

$$V^{(1)} = \langle x, x \rangle \quad \text{(the Hook potential)}, \quad F_i^{(1)} = x_i^2,$$

$$V^{(2)} = \langle Ax, x \rangle - V^{(1)}\langle x, x \rangle \quad \text{(the Garnier potential)}, \quad F_i^{(2)} = x_i^2(a_i - V^{(1)}),$$

$$V^{(3)} = \langle A^2 x, x \rangle - V^{(1)}\langle Ax, x \rangle - V^{(2)}\langle x, x \rangle, \quad F_i^{(3)} = x_i^2(a_i^2 - a_i V^{(1)} - V^{(2)})).$$

The rational and the Laurent polynomial solutions of (7.1) are given in [50, 27, 12, 13]. The basis for degrees -2 and -4 are given by

$$V_s^{(-1)} = \frac{1}{x_s^2} \quad \text{(the Rosochatius potentials)},$$

$$V_s^{(-2)}(x) = \frac{1}{x_s^4}\left(1 + \sum_{j\neq s}\frac{x_j^2}{a_s - a_j}\right), \quad s = 0,\ldots,n,$$

$$F_{s,i}^{(-1)} = \frac{1}{a_i - a_s}\frac{x_i^2}{x_s^2}, \quad F_{s,i}^{(-2)} = \frac{2}{a_i - a_s}\frac{x_i^2}{x_s^4}\left(1 + \sum_{j\neq s}\frac{x_j^2}{a_s - a_j}\right), \quad i \neq s,$$

$$F_{s,s}^{(-k)} = V_s^{(-k)} - \sum_{i\neq s}F_{s,i}^{(-k)}, \qquad k = 1,2.$$

7.2. Natural mechanical systems on ellipsoids with separable potentials. Consider the motion of a material point on an ellipsoid (1.1) under the influence of the potential

$$(7.4) \qquad V(x) = V^+(x) + \frac{1}{2}\sum_{i=0}^{n}\frac{\mu_i^2}{x_i^2}, \quad V^+(x) = \frac{1}{2}\sum_{k=1}^{m}\sigma_k V^{(k)}(x),$$

where σ_k are real parameters. The equations of motion are

$$(7.5) \qquad \dot{x} = y, \quad \dot{y} = -\frac{\langle A^{-1}y, y\rangle - \langle \nabla V(x), A^{-1}x\rangle}{\langle A^{-2}x, x\rangle}A^{-1}x - \nabla V(x).$$

As a straightforward generalization of Theorem 4.1, by using the constraints (1.4), the recurrence relations (7.3), and the identities

$$A_\lambda A^k = (\lambda - A)^{-1}A^k = \lambda^k A_\lambda^{-1} - \sum_{i=0}^{k-1}\lambda^{k-i}A^i, \qquad k \in \mathbb{N}$$

we get.

THEOREM 7.1. *Suppose that the eigenvalues a_i of the matrix A are distinct. Up to the action of the group $\mathbb{Z}_2^{n+1}$ generated by the reflections*

$$(x_i, y_i) \longmapsto (s_i x_i, s_i y_i), \quad s_i = \pm 1, \qquad i = 0, 1, \ldots, n,$$

the system (7.5) is equivalent to the matrix equation

$$(7.6) \qquad \dot{\mathcal{L}}(\lambda) = [\mathcal{L}(\lambda), \mathcal{A}(\lambda)]$$

with 2×2 matrices depending on the parameter λ

$$\mathcal{L}(\lambda) = \begin{pmatrix} q_\lambda(x,y) & q_\lambda(y,y) + q_\lambda(\frac{\mu}{x}, \frac{\mu}{x}) + \Delta(x,\lambda) \\ -1 - q_\lambda(x,x) & -q_\lambda(y,x) \end{pmatrix},$$

$$\mathcal{A}(\lambda) = \frac{1}{\langle A^{-2}x, x \rangle} \begin{pmatrix} 0 & \frac{1}{\lambda}(\langle \nabla V^+(x), A^{-1}x \rangle - \langle A^{-1}y, y \rangle - \langle A^{-1}\frac{\mu}{x}, \frac{\mu}{x} \rangle) - \Omega(x,\lambda) \\ \langle A^{-2}x, x \rangle & 0 \end{pmatrix},$$

where

$$\Delta(x,\lambda) = \sigma_1 \Delta_1(x,\lambda) + \cdots + \sigma_m \Delta_m(x,\lambda),$$
$$\Delta_k(x,\lambda) = \lambda^{k-1} - \lambda^{k-2}V^{(1)} - \lambda^{k-3}V^{(2)} - \cdots - \lambda V^{(k-2)} - V^{(k-1)},$$
$$\Omega(x,\lambda) = \langle A^{-2}x, x \rangle \left(\sigma_1 \Omega_1(x,\lambda) + \cdots + \sigma_m \Omega_m(x,\lambda) \right),$$

and $\Omega_k(x,\lambda)$ are determined from the equations

$$2\Omega_k(1 + q_\lambda(x,x)) = 2\Delta_k(x,\lambda) + \langle A_\lambda^{-1}x, \nabla V^{(k)}(x) \rangle, \qquad k = 1, \ldots, m.$$

For example,

$$\begin{aligned}
\Delta_1 &= 1, & \Omega_1 &= 1, \\
\Delta_2 &= \lambda - \langle x, x \rangle, & \Omega_2 &= \lambda - 2\langle x, x \rangle, \\
\Delta_3 &= \lambda^2 - \lambda\langle x, x \rangle - \langle Ax, x \rangle + \langle x, x \rangle^2, \\
\Omega_3 &= \lambda^2 - 2\lambda\langle x, x \rangle - 2\langle Ax, x \rangle + 3\langle x, x \rangle^2.
\end{aligned}$$

Now, the integrals (7.2) and the Lax representation are related by

$$\begin{aligned}
\det(\mathcal{L}(\lambda)) &= (1 + q_\lambda(x,x))\left(q_\lambda(y,y) + q_\lambda\left(\frac{\mu}{x}, \frac{\mu}{x} \right) + \Delta(x,\lambda) \right) - q_\lambda(x,y)^2 \\
&= \sum_{k=1}^m \lambda^{k-1}\sigma_k + \sum_{i=0}^n \frac{f_i}{\lambda - a_i} + \sum_{i=0}^n \frac{\mu_i^2}{(\lambda - a_i)^2}.
\end{aligned}$$

By taking $V^+ = \sigma/2 V^{(2)} = \sigma/2\langle Ax, x \rangle - \sigma/2\langle x, x \rangle^2$ and assuming $\mu_0 = 0$, in the limit $a_0 \to 0$, we get the Lax representation for a natural mechanical system in $\mathbb{R}^n$ under the influence of the the Garnier potential $V = \sigma/2\langle a\mathbf{x}, \mathbf{x} \rangle - \sigma/2\langle \mathbf{x}, \mathbf{x} \rangle^2$ obtained by Antonowicz and Rauch-Wojciechowski [2] (see also [41]). Note that, similarly as in Eilbecktt, Enol'skii, Kuznetsov and Tsiganov [17], one can consider the problem within a framework of r-matrix method.

Finally note that the polynomials $V^{(k)}$, as well as the Lax representation, are well defined for a symmetric ellipsoid (1.1), (5.1). Therefore, repeating the construction presented in Section 5, we obtain.

COROLLARY 7.1. *The system* (7.5) *on a symmetric ellipsoid* (1.1), (5.1) *is completely integrable in a non-commutative sense by means of integrals* (5.5) *and*

$$\tilde{f}_s = \sum_{i \in I_s} \left(y_i^2 + \sum_{k=1}^{m} \sigma_k F_i^{(k)} + \frac{\mu_i^2}{x_i^2} + \sum_{j \notin I_s} \frac{P_{ij}}{a_i - a_j} \right), \quad s = 0, 1, \ldots, r.$$

Generic trajectories take place over $r + \rho$-*dimensional invariant isotropic tori, spanned by the Hamiltonian vector fields* $X_{\tilde{f}_s}, X_{P_s}$. *Also, the problem is Liouville integrable by means of integrals* $\tilde{f}_s$ *and* (5.15).

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